J. Fluid Mech. (1973), *vol.* 60, *part* 4, *pp.* 733–749
Printed in Great Britain

Nonlinear resonant oscillations in open tubes

By BRIAN R. SEYMOUR
Department of Mathematics, University of British Columbia, Vancouver
AND MICHAEL P. MORTELL
Department of Mathematical Physics, University College, Cork, Ireland

(Received 25 November 1972)

A gas in a tube, one end of which is open, is driven by a periodic applied velocity or pressure at or near a resonant frequency. Damping is introduced into the system by radiation of energy through the open end. It is shown that shocks are possible at an open end and that there is a critical level of damping which ensures a continuous gas response for all frequencies. At the critical level the amplitude of the response is $O(\epsilon^{\frac{1}{3}})$, where ϵ is the amplitude of the input, and it is bounded by the amplitude predicted by linear theory. There is agreement with the qualitative experimental results available.

1. Introduction

A gas is contained in a tube which is open at one end. At the other end a periodic velocity or pressure drives the gas. In this paper we examine the small amplitude, periodic vibrations of the gas which result when the applied frequency is at or near a resonant frequency. Seymour & Mortell (1973; hereafter referred to as I) considered the case when the end of the tube was closed and showed how to introduce damping due to radiation into the theoretical model. The result was that the quantitative agreement between theory and experiment was quite satisfactory and considerably better than had been previously attained. Damping is introduced in the same manner here and one of the principal aims is to examine the effect of this damping on the frequency range for which no continuous solutions exist in the undamped theory. We prove, as was shown for the closed tube in I, that, for a given forcing function, there is a critical level of damping μ_c which ensures continuous solutions for all frequencies.

The basis for the analysis presented here is the realization that a sufficiently accurate estimate of the travel time of a wave over one cycle of the motion is essential for the prediction of a bounded motion at resonance. When the applied frequency is sufficiently far from the linear resonant frequency the travel time calculated from linear theory is adequate. However, as the difference between the linear travel time and the period of the applied signal tends to zero the nonlinear correction is no longer negligible and, in fact, is essential to ensure a bounded solution. Inclusion of the nonlinear terms results in amplitude dispersion which distorts the signal. This distortion must be cumulative over one period to affect the solution. We show that the distortion induced by the quadratic term in the equation of state (i.e. the first nonlinear correction to the travel time) is cumulative for reflexion from a closed end, but not for an open end. Hence, for an open end we require the second nonlinear correction to differentiate between travel times computed from linear theory and nonlinear theory. With an accurate travel time in hand, the boundary conditions lead to a nonlinear functional equation determining the signal carried by a wave. In the small rate limit this is reduced to a nonlinear ordinary differential equation.

The undamped limit of one of the problems (applied velocity) discussed here has been treated previously by van Wijngaarden (1968), Mortell (1971) and Collins (1971).† The results of van Wijngaarden's theory are that the pressure is continuous and has an amplitude $O(\epsilon^{\frac{1}{2}})$, where the piston amplitude is $O(\epsilon)$. The amplitude result is a direct consequence of the nonlinear boundary condition which he used to model the open end. An examination of his principal result, equation (7.8), shows that amplitude dispersion (nonlinear convection) plays no role in his model. The theory used by Mortell (1971) is based on a travel time which depends only on terms up to second order in the equation of state. For reasons stated above, and expanded upon in the text, this is not a sufficiently accurate model. Collins (1971) showed that, under certain restrictions on the applied frequency, the undamped gas motion was continuous with an amplitude $O(\epsilon^{\frac{1}{3}})$. His technique for deriving the governing differential equation is essentially a perturbation expansion in powers of the *resulting* amplitude $\epsilon^{\frac{1}{3}}$. The disadvantage of using this procedure for the damped motion is that the order of magnitude of the resulting motion in terms of fractional powers of ϵ must be assumed. Since this is determined by the level of damping present, it is more useful to obtain a governing equation valid for a range of the damping parameter and hence a variety of resulting amplitudes. We find such an equation here by calculating the travel time using a variant of Lin's technique (1954).

Damping is introduced into the theory through the boundary condition $p(t) = -ju(t)$ at $x = 0$, where p and u are suitably normalized pressure and velocity. The case $j = 0$ corresponds to an open end in acoustic theory. When $0 < j \ll 1$, there is loss of energy through the boundary since the transmission coefficient $\mu = 2j/(1+j)$ is non-zero. We prove that when $\mu > \mu_c = O(\epsilon^{\frac{2}{3}})$ a continuous solution exists for all frequencies. When $\mu < \mu_c$ there is a well-defined frequency band in which continuous solutions are not possible. Outside this band there is again a continuous periodic solution. Whenever there is a continuous solution it is unique, and is bounded by the linear solution which has magnitude $O(\epsilon/\mu)$.

The theoretical work of van Wijngaarden is substantiated in his paper by his experimental observations. B. Sturtevant (1972, private communication) has made available to us certain features of experiments which he has conducted using a piston to drive a gas in an open tube. He reports that the pressure on the piston is continuous and has an amplitude which is consistent with $O(\epsilon^{\frac{1}{3}})$. The pressure at the open end may be discontinuous. Lettau (1939) observed shocks at the open end when the piston operated at a higher harmonic.

† See *Note added in proof*, p. 748.

2. Formulation linear theory

A column of gas, of length L in some reference state, is contained in a pipe. One end of the pipe is open while at the other end a periodic applied velocity or traction drives the gas. If the pressure and density are measured from their values in the reference state (p_0, ρ_0) with the associated sound speed a_0, then in terms of the non-dimensional variables $a_0 u$, $\rho_0 a_0 p$, $\rho_0 \rho$, and Lx and $La_0^{-1} t$, the governing equations in Lagrangian form are

$$[(1+e)^{-1}]_t - u_x = 0 \tag{2.1}$$

and

$$u_t + p_x = 0, \tag{2.2}$$

where $e\,(=\rho-1)$ is the condensation, γp the excess pressure ratio and u the non-dimensional particle velocity. The equation of state for the isentropic flow of an ideal gas in these variables is

$$\gamma p = (1+e)^{\gamma} - 1. \tag{2.3}$$

The end $x = 0$ is open or 'nearly open' in the sense that the possibility of radiation of energy through this end is allowed. Across the interface at $x = 0$ the pressure and velocity are continuous, and so the disturbance must be compatible with the homogeneous boundary condition

$$p(0,t) = -ju(0,t), \tag{2.4}$$

where $(\gamma j)^{-1}\ (\geqslant 0)$ is the impedance of the interface. The essential assumption in writing (2.4) is that the motion in the surrounding medium is generated by the passage of a simple wave. A fuller discussion of such a boundary condition is given by Mortell & Varley (1970). We consider two separate boundary conditions at $x = 1$ which, it turns out, produce similar vibrations in the gas. They are

$$u(1,t) = H(\omega t) \quad (u, p \text{ problem})$$

and

$$p(1,t) = H(\omega t) \quad (p, p \text{ problem}). \tag{2.5}$$

The u, p problem corresponds to a gas in an open-ended tube being driven by an applied piston velocity at the other end. The p, p problem corresponds to a gas motion generated by an applied pressure or traction at one end while the other end is open. The amplitude of H is $\epsilon\ (\ll 1)$ and its period is normalized so that $H(y+1) = H(y)$. Further, H has zero mean over one period, so that

$$\int_0^1 H(s)\, ds = 0. \tag{2.6}$$

Equations (2.1)–(2.3) are nonlinear and admit discontinuous solutions. However, it has been shown by Mortell & Seymour (1973) that for *any* time-periodic motion the mean pressure and velocity do not vary from particle to particle. If the constant mean pressure and velocity of the periodic state are chosen as the reference state, (2.4)–(2.6) imply that the means of u and p are zero. We seek solutions u and p of (2.1)–(2.5) which have the same period as H and have zero mean over this period.

A boundary condition of the form (2.4) was used in I in discussing motions in a nearly closed tube ($j \gg 1$) generated by an applied velocity (the u, u problem). Two significant consequences were noted: the introduction of damping through (2.4) could prevent the occurrence of shocks in the flow and always reduced the amplitude of the motion. The latter result was used to obtain a better correlation between theory and experiment by an appropriate choice of j. The parameter j will be used in the same way here. A discussion of j as a 'lumped' damping parameter is also given in I.

2.1. *Linear theory*

When (2.1)–(2.3) are linearized the pressure and velocity are given by

$$p = -f(\eta+\omega x)-g(\eta-\omega x), \quad u = f(\eta+\omega x)-g(\eta-\omega x), \tag{2.7}$$

where $\eta = \omega t$ and f and g are arbitrary. The boundary condition (2.4) implies that $g(\eta) = -kf(\eta)$, where $k = (1-j)/(1+j)$ is the *reflexion coefficient* at $x = 0$. For the p, p problem ($k = 1$), on applying (2.5) and eliminating g, f satisfies

$$f(\eta+\omega)-f(\eta-\omega) = -H(\eta). \tag{2.8}$$

This is a *linear functional equation* which relates the signal f on $x = 1$ to the known forcing function H. We seek solutions of (2.8) which, like H, have unit period. Further, as a consequence of (2.7) and the fact that u and p have zero mean over one cycle, f and g also satisfy

$$\int_0^1 f(s)\,ds = \int_0^1 g(s)\,ds = 0. \tag{2.9}$$

Equation (2.8), which determines f for the p, p problem, has no bounded solutions with unit period when $\omega = \omega_0 = \frac{1}{2}n$ ($n = 1, 2, 3, \ldots$). Similarly, the corresponding functional equation which determines f for the u, p problem has no bounded solutions with unit period when $\omega = \omega_0 = \frac{1}{4}n$ ($n = 1, 3, 5, \ldots$). The frequencies ω_0 are the *linear resonant* frequencies. In this paper we construct a nonlinear theory which allows a bounded solution with unit period for any frequency and examine the effect of damping on these solutions.

Linear theory fails when the travel time of the waves coincides with the period of the driver. Then all amplitudes in the signal are simultaneously in phase with the driver. A nonlinear wave is amplitude dispersed, so that different amplitudes in the signal travel with different speeds. Only wavelets of a particular amplitude can be in phase with the driver, while all others are out of phase. For example, when the resonating wavelets are those carrying zero amplitude (and hence travelling at the linear speed) the system is operating at a linear resonant frequency. The nonlinear contribution to the travel time in the tube is always only a correction. However, as the difference between the linear travel time and the period of the driver tends to zero, the nonlinear contribution cannot be neglected – in fact it will be the dominant part of the functional equation. It is clear that a suitably accurate approximation to the nonlinear travel time is needed to solve the resonant problem. A systematic means of doing this is given in the next section.

3. Representations for the travel time

In this section a variant of Lin's (1954) technique is used to find systematically the influence of nonlinearity on the travel time of waves in the tube. This technique was used in Mortell (1971) to calculate the first correction to the linear travel time. It emerges here that resonant phenomena involving open ends depend on the third-order terms, and thus the second correction to the linear travel time is needed.

It is convenient to reformulate the basic equations (2.1)–(2.3) in terms of their Riemann invariants and characteristic curves. Upon defining

$$c(e) = \int_0^e a(s)\,(1+s)^{-1}\,ds, \tag{3.1}$$

where

$$a^2(e) = (1+e)^2\,(dp/de), \tag{3.2}$$

(2.1)–(2.3) define the Riemann invariants

$$\left.\begin{aligned} 2f(\beta) &= u-c = u-p+O(e^2),\\ \text{and}\qquad -2g(\alpha) &= u+c = u+p+O(e^2).\end{aligned}\right\} \tag{3.3}$$

The associated characteristics α and β are given by

$$\frac{dx}{d\beta} = a(c)\frac{dt}{d\beta},\quad \frac{dx}{d\alpha} = -a(c)\frac{dt}{d\alpha}. \tag{3.4}$$

For the isentropic flow of an ideal gas, (3.1) and (3.2) imply that

$$a(c) = 1+Mc+Nc^2+O(c^3), \tag{3.5}$$

where $M = \frac{1}{2}(\gamma+1) > 0$ and $N = \frac{1}{8}(\gamma-1)(\gamma-2)$.

We assume an expansion of the form

$$\left.\begin{aligned} x(\alpha,\beta) &= x_0(\alpha,\beta)+x_1(\alpha,\beta)+x_2(\alpha,\beta)+\ldots,\\ \text{and}\qquad t(\alpha,\beta) &= t_0(\alpha,\beta)+t_1(\alpha,\beta)+t_2(\alpha,\beta)+\ldots,\end{aligned}\right\} \tag{3.6}$$

where $t_i(\alpha,\beta)$ and $x_i(\alpha,\beta)$ are $O(f^i, g^i)$. Then (3.4) implies that

$$x_{0\beta} = t_{0\beta},\quad x_{0\alpha} = -t_{0\alpha}.$$

Thus

$$\alpha = \omega(t_0-x_0),\quad \beta = \omega(t_0+x_0),$$

where α and β have been parametrized by the condition that

$$\alpha = \beta = \omega t \quad \text{on} \quad x = 0.$$

On using (3.5) and (3.3) and noting $\omega t_{0\alpha} = \omega t_{0\beta} = \frac{1}{2}$, (3.4) gives, at the next order,

$$\omega(x_1-t_1) = \psi_1(\alpha)-\tfrac{1}{2}M\left[\beta g(\alpha)+\int_\alpha^\beta f\,ds\right],$$

and

$$\omega(x_1+t_1) = \phi_1(\beta)+\tfrac{1}{2}M\left[\alpha f(\beta)-\int_\alpha^\beta g\,ds\right].$$

The functions ψ_1 and ϕ_1 are determined by the condition that on $x_1 = 0$

$$\alpha/\omega = t_0 = t, \quad \beta/\omega = t_0 = t.$$

Then t and x are given, to this order, by

$$\left.\begin{aligned} \omega(t-x) &= \alpha + \tfrac{1}{2}M\left[(\beta-\alpha)\,g(\alpha) + \int_\alpha^\beta f(s)\,ds\right], \\ \text{and} \quad \omega(t+x) &= \beta - \tfrac{1}{2}M\left[(\beta-\alpha)f(\beta) + \int_\alpha^\beta g(s)\,ds\right]. \end{aligned}\right\} \tag{3.7}$$

The calculation of the next term is lengthy but follows the same pattern. The final expressions are

$$\begin{aligned} \omega(t_2-x_2) = \tfrac{1}{4}M^2\Big[g(\alpha)\int_\alpha^\beta (2f-g)\,ds + (\beta-\alpha)\,g^2(\alpha) - (\beta-\alpha)f(\beta)\,g(\alpha) \\ - \int_\alpha^\beta fg\,ds - \tfrac{1}{2}(\beta-\alpha)f^2(\beta) + \frac{1}{2}\int_\alpha^\beta f^2 ds\Big] \\ - \tfrac{1}{2}N\left[(\beta-\alpha)\,g^2(\alpha) + \int_\alpha^\beta f^2 ds + 2g(\alpha)\int_\alpha^\beta f\,ds\right], \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} \omega(t_2+x_2) = -\tfrac{1}{4}M^2\Big[f(\beta)\int_\alpha^\beta (2g-f)\,ds + (\beta-\alpha)f^2(\beta) - (\beta-\alpha)f(\beta)\,g(\alpha) \\ - \int_\alpha^\beta fg\,ds - \tfrac{1}{2}(\beta-\alpha)\,g^2(\alpha) + \frac{1}{2}\int_\alpha^\beta g^2 ds\Big] \\ + \tfrac{1}{2}N\left[(\beta-\alpha)f^2(\beta) + \int_\alpha^\beta g^2 ds + 2f(\beta)\int_\alpha^\beta g\,ds\right]. \end{aligned} \tag{3.9}$$

The travel time T for one traversal down and back in the tube is given by

$$\begin{aligned} T = [\omega^{-1}(\beta-\alpha)]_{x=1} = 2 + \frac{M}{2\omega}\left[(\beta-\alpha)(f(\beta)+g(\alpha)) + \int_\beta^\beta (f+g)\,ds\right] \\ - \frac{M^2}{4\omega}\Big[f(\beta)\int_\alpha^\beta (2g-f)\,ds + g(\alpha)\int_\alpha^\beta (2f-g)\,ds + \tfrac{1}{2}(\beta-\alpha)(f^2(\beta)+g^2(\alpha)) \\ - 2(\beta-\alpha)f(\beta)\,g(\alpha) - 2\int_\alpha^\beta fg\,ds + \frac{1}{2}\int_\alpha^\beta (f^2+g^2)\,ds\Big] \\ - \frac{N}{2\omega}\Big[(\beta-\alpha)(f^2(\beta)+g^2(\alpha)) + \int_\alpha^\beta (f^2+g^2)\,ds \\ + 2f(\beta)\int_\alpha^\beta g\,ds + 2g(\alpha)\int_\alpha^\beta f\,ds\Big], \end{aligned} \tag{3.10}$$

when use is made of (3.7)–(3.9). It is noted that no appeal to the boundary conditions has been made in deriving (3.10).

If now we consider the u, u problem in which the end $x = 0$ is closed, so that $g = f$, and $u = H$ on $x = 1$, then (3.10) implies that

$$T = 2 + (M/\omega)(\beta-\alpha)f(\alpha) + O(f^2, H). \tag{3.11}$$

The integral term in (3.10) has vanished owing to the zero-mean condition (2.9). In this case $f = O(\epsilon^{\frac{1}{2}})$ (see I) and then the correction terms f^2 and H are of the same order.

On the other hand, if $x = 0$ is open, so that $g = -f$, and $p = H$ on $x = 1$, then (3.10) implies that

$$T = 2 + O(f^2, H). \tag{3.12}$$

The result (3.12) shows that if one uses only the *first* nonlinear correction, given by (3.7), to the characteristics to solve a problem involving an open end, then the signal f is determined by a *linear* functional equation. This point was not appreciated in Mortell (1971). In the analysis given there for problems involving an open end the functional equation was derived on the basis of (3.7) to model reflexion from a piston and it was mistakenly assumed that the distortion would further accumulate on reflexion from the open end.

The basic difference between resonant motions in open and closed tubes is typified by (3.11) and (3.12). The correction to the linear term at order f in (3.11) shows that there is cumulative distortion of the signal at this order for reflexion at a closed end. The cumulative distortion does not occur until order f^2 for reflexion from an open end, and consequently the travel time must be calculated to this order.

4. Functional and differential equations for the signals

In this section we set up functional equations, on the basis of (3.10), for both the u, p and p, p problems when the end $x = 0$ is 'nearly' open. These are then reduced, in the small rate limit, to nonlinear ordinary differential equations for the signal f on $x = 1$. For simplicity the details for the p, p problem only are given.

Consider the wavelet $\beta = \beta_0$ leaving $x = 1$ at time $t = t_0$ which is reflected at $x = 0$ at time $t = \beta_0 = \alpha_1$. This reflected wavelet $\alpha = \alpha_1$ arrives at $x = 1$ at time t_1, and is reflected as $\beta = \beta_1$ (see figure 1). The boundary condition (2.4) implies that

$$g(\alpha_1) = -kf(\beta_0), \quad k = (1-j)/(1+j), \tag{4.1}$$

while the condition that, on $x = 1$, $p = H(\omega t)$ yields

$$f(\beta_1) + g(\alpha_1) = -H(\omega t_1). \tag{4.2}$$

Elimination of g gives

$$f(\beta_1) - kf(\beta_0) = -H(\omega t_1), \tag{4.3}$$

where

$$\beta_1 = \beta_0 + \omega T. \tag{4.4}$$

T is given by (3.10), in which β is replaced by β_1, α by β_0 and g by $-kf$. Equations (4.3), (4.4) and (3.10) then define a nonlinear functional equation for the unknown signal f. The expression (3.10), however, is unmanageable in its present form. We simplify it as follows. First, we set $\omega = \frac{1}{2}n(1+\delta)$; that is, we are interested in frequencies near the linear resonant frequency $\omega_0 = \frac{1}{2}n$. Then $\omega^{-1}(\beta - \alpha)$ is approximated by $2 + M[f(\beta) - kf(\alpha)]$ for terms involving f and by 2 for terms involving f^2. The integrals are treated in the same way. Thus the error always

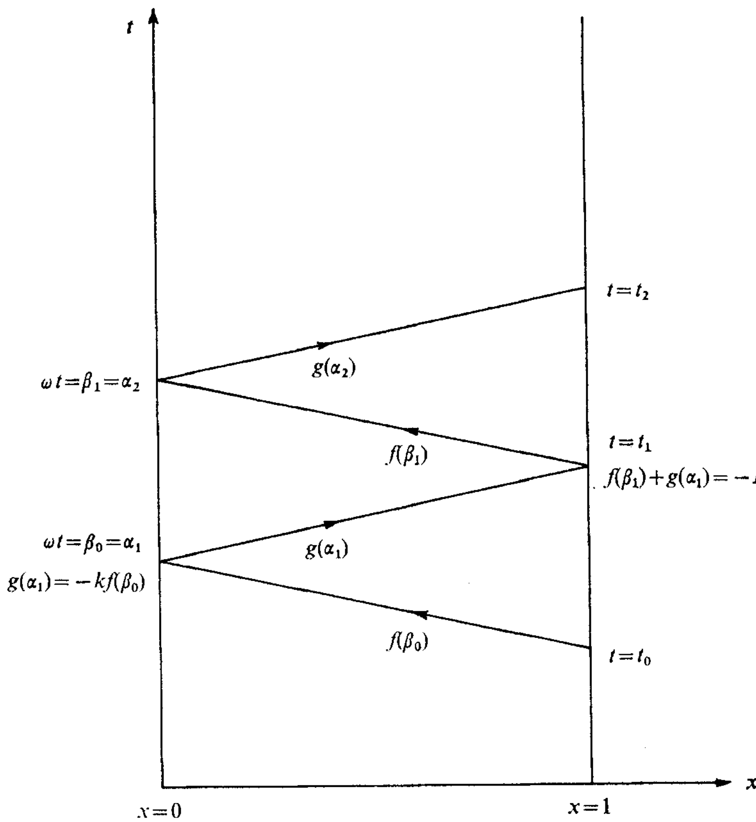

FIGURE 1. Wave diagram for p, p problem.

remains $O(f^3)$. When corresponding terms are then grouped, (3.10) yields, on using (4.3), the zero-mean condition (2.9) and $|1-k| \ll 1$ ('nearly' open),

$$\beta_1 = \beta_0+n+n\delta+M[-\omega H(\omega t_1)+\tfrac{1}{2}n\delta f(\beta_0)] + \left[\tfrac{1}{2}\omega(5M^2-4N)f^2(\beta_0)+(\tfrac{3}{4}M^2-N)\int_{\beta_1-n}^{\beta_1} f^2(s)\,ds\right]. \quad (4.5)$$

In the *small rate limit* $|\omega f'| \ll 1$ and $|n\delta| \ll 1$, if $f(\beta_1)$ is expanded using (4.5) and the periodicity condition $f(\eta) = f(\eta+n)$, then (4.3) and (4.4) reduce to

$$[n(1+\delta)bf^2(\eta)-\omega MH(\eta)+\tfrac{1}{2}n\delta Mf(\eta)+n(\delta+a)]f'(\eta) = -\bar{H}(\eta)-\mu f(\eta), \quad (4.6)$$

where $\bar{H}(\eta) = H(\eta+\omega), \quad b = \tfrac{1}{4}(5M^2-4N), \quad \mu = 1-k$

and $$na = (\tfrac{3}{4}M^2-N)\int_{t-n}^{t} f^2(s)\,ds > 0.$$

The order of magnitude of f depends on the size of the transmission coefficient μ, where $\mu \ll 1$. Linear theory implies that always $f = O(H/\mu)$. To obtain a consistent balance of terms in (4.7) nonlinear theory indicates that $f = O(H^r)$, where $0 < r < 1$. Then in all cases (4.6) may be replaced by

$$n[(1+\delta)bf^2+\delta+a]f' = -\bar{H}-\mu f. \tag{4.7}$$

Equation (4.7) determines the periodic signal function f at the boundary $x = 1$.

It is evident from the expression (3.10) that there is an interaction between the waves travelling in opposite directions. The α and β waves are non-interacting only if the trajectory of an α wave is unaffected by the β waves through which it passes, and vice versa. A calculation formally neglecting any interaction would decrease the value of b slightly and set a equal to zero. The parameter a (> 0) is a constant, for a given signal f, and is proportional to the mean of f^2 over one period. Thus the interaction of the waves increases the travel time of all wavelets by a constant and, accordingly, decreases the resonant frequencies by a corresponding amount. Consequently resonant peaks occur to the left of that predicted by the linear theory discussed above. If the interaction is neglected, the response curves must be translated along the frequency axis by an amount of order f^2.

The u, p problem is treated in the same manner as the above. In this case $\omega = \frac{1}{4}n(1+\delta)$, where $\omega_0 = \frac{1}{4}n$ ($n = 1, 3, 5, \ldots$), are the linear resonant frequencies. The calculation is longer as expressions for both $\beta_1-\beta_0$ and $\beta_2-\beta_1$ must be found. This results because now two traversals of the tube correspond to one period of the motion. The final equation satisfied by f on $x = 1$ is again (4.7), where $\mu = 2(1-k)$ and $-\bar{H}(\eta)$ is replaced by $H(\eta+\omega)-kH(\eta-\omega)$.

5. Analysis of differential equation

In this section we analyse the integral curves of (4.7) for various ranges of the parameters μ and δ. The transmission coefficient μ introduces damping into the system, while δ measures the deviation from the linear resonant frequency ω_0. The integral curves are used to construct the signal f on $x = 1$, which must additionally satisfy the zero-mean condition. To ensure that f is single valued, it may be discontinuous. This is discussed further in § 6.

We first distinguish between the integral curves of (4.7) and the signal f which further satisfies the zero-mean condition. An integral curve is denoted by $Z(\eta)$. While f is defined only for $0 \leqslant \eta \leqslant 1$ and is then continued periodically, if f is continuous and $f(0) = f(1)$, f must coincide with an integral curve Z which is both continuous and periodic for all η. Conversely, a continuous periodic integral curve Z which satisfies the mean condition is the required signal f. When such a curve exists it is unique. When no such curve exists, f is discontinuous and is composed of distinct integral curves. We therefore seek a periodic solution f of

$$n[(1+\delta)bZ^2+\delta+a]Z' = G(\eta)-\mu Z, \tag{5.1}$$

where $a, b, \mu \geqslant 0$, G is a periodic function such that

$$\int_0^1 G(s)\,ds = 0$$

and we require that

$$\int_0^1 f(s)\,ds = 0.$$

Integrating (5.1) we obtain, for a continuous solution curve Z,

$$\tfrac{1}{3}nb(1+\delta)\,[Z^3(\eta)-Z^3(0)]+n(\delta+a)\,[Z(\eta)-Z(0)] = \int_0^\eta [G(s)-\mu Z(s)]\,ds. \quad (5.2)$$

When $\mu = 0$ equation (5.2) implies that all continuous solutions are periodic. In the damped case $\mu > 0$, (5.2) implies that a continuous solution with zero mean is periodic and conversely. The existence of a solution with zero mean is easily proved by appealing to the continuous dependence of the solutions on the initial data $Z(0)$.

Case $\delta > -a, \mu \geqslant 0$

When $\delta > -a$ equation (5.1) implies that all integral curves have bounded derivatives for all η. There exists a unique periodic solution $f(\eta)$ with zero mean. When $\mu > 0$ this is the only periodic solution. Since the curve $Z_l = G(\eta)/\mu$ is the isocline $Z' = 0$ this solution is bounded by the linear solution Z_l.

For $\mu = 0$ equation (5.2) becomes a cubic in $Z(\eta)$ where all integral curves have unit period. This is the solution given by Collins (1971), who, for the u, p problem, considered only the case $\mu = 0$, $\delta > -a$.

Case $\delta < -a, \mu = 0$

When $\delta < -a$ and $\mu = 0$ equation (5.1) has singular points where both $G(\eta) = 0$ and $Z(\eta) = \pm\phi(\delta)$, where $\phi^2(\delta) = -[(\delta+\alpha)/(1+\delta)]/b > 0$. If the zeros of $G(\eta)$ are at $\eta = \eta_i$ ($i = 1, 2, \ldots$), then in the η, Z plane the singular points are $(\eta_i, \pm\phi)$. The points $(\eta_i, \pm\phi)$ are saddles when $G'(\eta_i) \gtrless 0$ and centres when $G'(\eta_i) \lesseqgtr 0$. The only solution curves which are single valued are those for which $Z > \phi$ or $Z < -\phi$ for all η. All other curves pass through both $Z = \pm\phi$ and are multivalued. There is consequently no single-valued integral curve with zero mean value. All the solution curves passing through $Z = 0$ have magnitude $O(\phi)$. As $\delta \to -1$, $\phi \to \infty$, which indicates that the nonlinear undamped model is not realistic for $\delta < -a$. It is to be expected that, as the frequency moves away from ω_0, the linear solution is recovered. This is not implied by the nonlinear equation (5.1). However, when $\delta = O(\epsilon^{\frac{2}{3}})$ a consistent matching solution is obtained from the appropriate linear equation, i.e. (5.1) with $a = b = 0$. This yields a continuous solution f of order $\epsilon^{\frac{1}{3}}$.

Case $\delta < -a, \mu > 0$

We consider the case $0 < \mu \ll 1$ so that we are dealing with a resonant motion subject to a small amount of damping. As for the u, u problem (see I), we show that for a given forcing function there is a critical amount of damping μ_1 such that for $\mu > \mu_1$ the signal f is continuous for all frequencies. For $\mu < \mu_1$ there is a frequency band in which f may be discontinuous.

The positions of the singular points of (5.1) now depend on both μ and δ. Since they are located at the intersections of the lines $Z = \pm\phi(\delta)$ and the curve $Z = G(\eta)/\mu$ their number will depend on the shape of G. For any G, since $\phi \to \infty$ as $\delta \to -1$, there exists a $\delta_c < -a$ such that, when $\delta < \delta_c$, $\phi > \max_\eta |G/\mu|$. Then there are no singular points, but the lines $Z = \pm\phi$ are the isoclines $|Z'| = \infty$. δ_c is defined by

$$\phi(\delta_c) = \max_\eta |G/\mu| = G_m/\mu,$$

so that

$$\delta_c = -[(a+h)/(1+h)], \tag{5.3}$$

where $h = b(G_m/\mu)^2 > 0$. The existence of a unique periodic solution with zero mean is not obvious in this case ($\delta < \delta_c$), since any curve crossing $Z = \pm\phi$ will be multi-valued. It is proved in appendix A that such a solution curve $f(\eta)$ does exist for $\delta < \delta_c$ and that f is bounded by the linear solution $Z_l = G(\eta)/\mu$.

Case $\delta_c < \delta < -a$

We have shown the existence of a unique, continuous, periodic solution with zero mean for the frequencies $\delta > -a$ and $\delta < \delta_c < -a$. In both of these ranges f is bounded by the linear solution. For the range $\delta_c < \delta < -a$ the signal function may not be continuous.

When $\delta_c < \delta < -a$ there exist isolated singular points which are alternately saddles and either nodes or spirals. The possible values of $Z'(\eta)$ at these points are

$$\lambda = [-\mu \pm (I^{\pm})^{\frac{1}{2}}]/\pm 4nb(1+\delta)\phi, \tag{5.4}$$

where

$$I^{\pm}(\eta) = \mu^2 + 8nb(1+\delta)\phi G'(\eta). \tag{5.5}$$

Hence they are classified as follows: at the intersection of $Z = G/\mu$ and $Z = \phi\,[-\phi]$ there is (i) a saddle point if $G'(\eta) > 0$ [< 0]; (ii) a node if $G'(\eta) < 0$ [> 0] and $I^+(\eta) > 0$ [$I^-(\eta) > 0$]; (iii) a spiral if $G'(\eta) < 0$ [> 0] and $I^+(\eta) < 0$ [$I^-(\eta) < 0$]. There is always an even number of singular points per period, the number being determined by the magnitude of ϕ and the number of relative maxima and minima of G. We label the saddle points $A_0 A_2, \ldots$ and the node/spirals as $B_1, B_3, \ldots$, where $A_0, B_1, A_4, \ldots$ lie on $Z = \phi$ and $A_2, B_3, A_6, \ldots$ lie on $Z = -\phi$. There is always the possibility that a separatrix connects the points A_n, B_{n+1} and A_{n+2}. If this happens for all n the resulting separatrix is both continuous and periodic (see figure 2) and hence, by (5.2), has zero mean. Obviously the nodal conditions, $I^+(\eta) > 0$ at $B_1, B_5, \ldots$ and $I^-(\eta) > 0$ at $B_3, B_7, \ldots$, are *necessary* for the existence of such a solution. It is proved in appendix B that

$$\left.\begin{aligned} I^+(\eta) > 0 \quad &\text{for} \quad A_i < \eta < A_{i+2} \quad (i = 0, 4, \ldots),\\ \text{and} \qquad I^-(\eta) < 0 \quad &\text{for} \quad A_{i+2} < \eta < A_{i+4} \quad (i = 0, 4, \ldots)\end{aligned}\right\} \tag{5.6}$$

is a *sufficient* condition for the existence of a unique, continuous, periodic solution with zero mean for a particular frequency $\delta_c < \delta < -a$. If $G'_m = \max_\eta |G'(\eta)|$, (5.5) implies that condition (5.6) can be written as

$$(\delta - \delta^-(\mu))\,(\delta - \delta^+(\mu)) > 0, \tag{5.7}$$

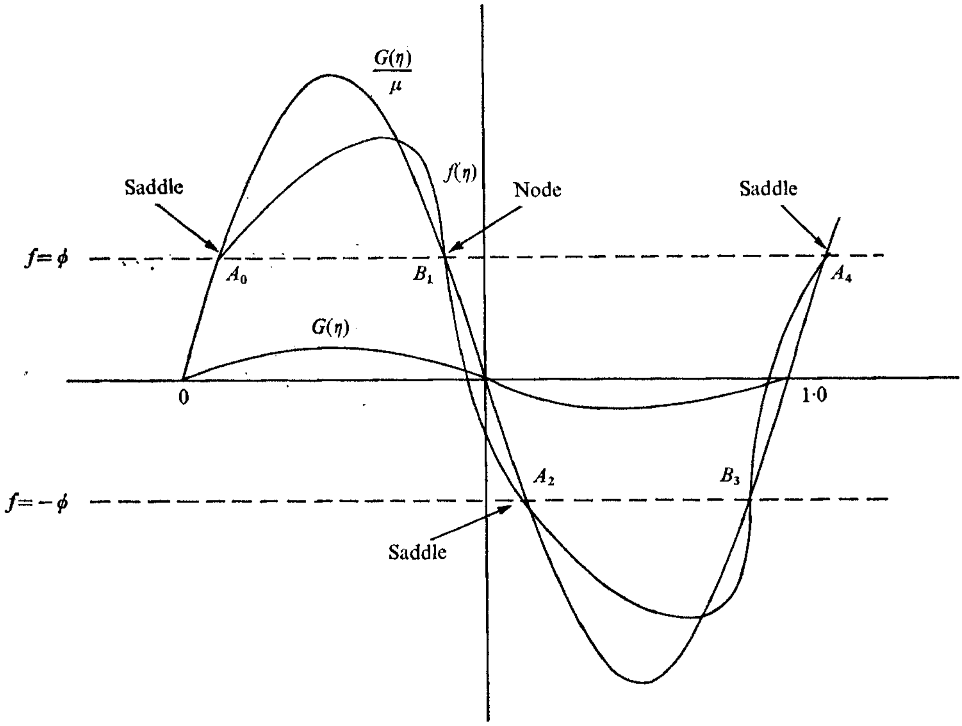

FIGURE 2. Sketch of solution curve $f(\eta)$ when $\mu > \mu_c$.

where
$$\delta^{\pm}(\mu) = -\tfrac{1}{2}\{1+a \mp [(1-a)^2 - 4\mu^4(8nG'_m\sqrt{b})^{-2}]^{\frac{1}{2}}\}. \tag{5.8}$$

Note that $-1 = \delta^-(0) \leqslant \delta^- \leqslant \delta^+ \leqslant \delta^+(0) = -a$ and that (5.6) is satisfied for all frequencies provided that the roots of (5.7) are complex. Thus for a given forcing function $G(\eta)$ there is a continuous periodic solution for *any* δ when $\mu > \mu_1 > 0$, where μ_1 is given by
$$\mu_1^2 = (1-a)\,8nG'_m\sqrt{b}, \tag{5.9}$$
and also for $\delta > \delta^+(\mu)$ or $\delta < \delta^-(\mu)$ when $0 < \mu < \mu_1$. Equation (5.9) implies that $\mu_1 = O(\epsilon^{\frac{1}{2}})$. We show in § 6 that there exists a $\mu_c < \mu_1$ with the same property. When there is a spiral at any B_i there is no continuous periodic solution.

Case $|\delta + a| \leqslant O(\epsilon),\ \mu \geqslant 0$

Equation (5.1) implies that, when $|\delta+a| \leqslant O(\epsilon)$, $|Z'| \geqslant O(1)$ whenever $Z \leqslant O(\epsilon)$. Thus there is a frequency band of width $O(\epsilon)$ about $\delta = -a$ for which, even though the solution may be continuous, the small rate condition is violated. The signal on the piston is then determined by a nonlinear functional equation, defined by (4.3), (4.4) and (3.10), rather than the differential equation (5.1). Notice that, for $|\delta| \ll 1, a = a(\mu)$.

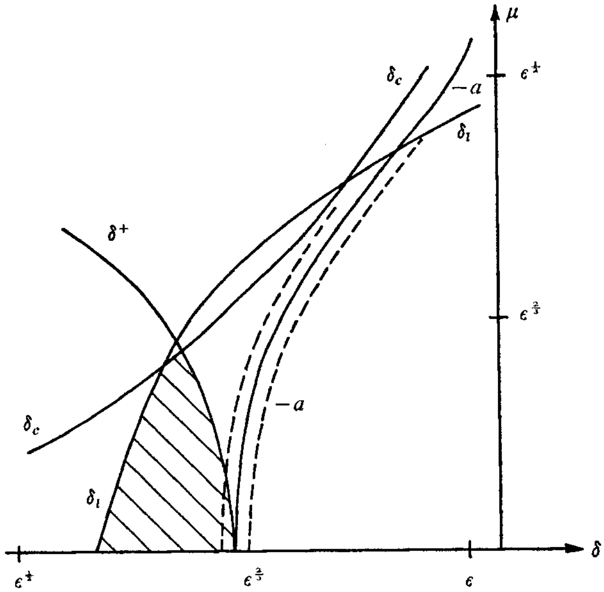

FIGURE 3. μ, δ plane. No continuous solutions when parameters in shaded area. Breaking length of wave less than tube length when the parameters lie between dashed lines.

6. Discussion

Here we interpret the results of § 5 in terms of the regions in the μ, δ plane where the solution curve is continuous or may be discontinuous. The qualitative results are compared with experiment.

The important critical frequencies are $\delta_c(\mu)$, defined by (5.3), and $\delta^{\pm}(\mu)$, defined by (5.8). Since $\delta^- < \delta_c$, for $|\delta| \ll 1$, the boundaries of the region in the μ, δ plane for which continuous solutions are possible are determined by δ_c and δ^+. The analysis of § 5 then implies that, for $\mu < \mu_1$, f may be discontinuous for $\delta_c < \delta < \delta^+$. Equations (5.3) and (5.8) imply that the curves $\delta_c(\mu)$ and $\delta^+(\mu)$ intersect when

$$\mu^3 + bG_m^2\mu = 8(1-a)\,bnG'_m G_m. \tag{6.1}$$

Equation (6.1) has only one real root, which, to lowest order in ϵ, is

$$\mu_c = [8(1-a)\,bnG'_m G_m]^{\frac{1}{3}}. \tag{6.2}$$

For $n = O(1)$, $\mu_c = O(\epsilon^{\frac{2}{3}}) < \mu_1$. When $\mu > \mu_c$ the solution f is continuous for all frequencies. When $\mu < \mu_c$ the solution is continuous for $\delta < \delta_c$ and for $\delta > \delta^+$. However, since $\delta_c \to -\infty$ as $\mu \to 0$, there is a more realistic lower edge of the 'discontinuous region' given by $\delta = \delta_l(\mu)$, where δ_l is the frequency from the corresponding linear system which yields a consistent matching of amplitudes with the nonlinear solution. For $0 \leqslant \mu \leqslant O(\epsilon^{\frac{2}{3}})$, δ_l is $O(\epsilon^{\frac{2}{3}})$. Thus the 'discontinuous region' is defined by $\mu < \mu_c$ and $\delta_m < \delta < \delta^+$, where $\delta_m = \max(\delta_l, \delta_c)$. In this region the theory breaks down, for it predicts one or more discontinuities in pressure at the piston per period. In most cases there will be two discontinuities of opposite signs. This is physically unacceptable as it corresponds to both compression and rarefaction shocks in the gas. In figure 3 we sketch the critical frequencies δ_c, δ^+, δ_l and $-a$ as functions of μ measured on a scale of powers of ϵ.

The experimental results of Sturtevant (1972, private communication) are that there is a frequency band near the fundamental linear resonant frequency in which the pressure *on the piston* is continuous, but contains possible discontinuities of slope. The presence of shocks at the open end was first observed and reported by Lettau (1939) when the piston was operating at a multiple of the fundamental frequency. For the piston amplitudes used by Sturtevant, shocks appeared at the open end at the fundamental frequency. These observations indicate two properties of the system. The first is that, if the boundary condition (2.4) is used to model conditions at an open end, the transmission coefficient μ is non-zero and at least $O(\epsilon^{\frac{2}{3}})$. The second is that, for a range of frequencies, the 'breaking length' of a wave (i.e. the minimum distance it takes for a shock to form) is less than the length of the tube. Since the 'breaking length' is proportional to $[\max_{0\leqslant\eta\leqslant 1} f'(\eta)]^{-1}$ this indicates that for certain frequencies $f' \geqslant O(1)$. The results of §5 predict that this is the case for $|\delta+a| \leqslant O(\epsilon)$. Since there is significant distortion in one traversal of the tube the small rate theory of §4 is invalid and the *nonlinear* characteristics must be used to describe the propagation of a wave. Then a shock forming between the piston and the open end is reflected as a rarefaction wave with a discontinuous slope. Since these 'shock frequencies' are only a small part of the frequency band in which the gas motion is amplified, it is not expected that shock dissipation is the primary attenuating mechanism. This is supported by a consideration of the balance of energy. Since the input energy at $x = 1$ is $O(\epsilon f)$, shock dissipation $O(f^3)$ and radiation loss $O(\mu f^2)$, a balance dominated by shock dissipation would yield $f = O(\epsilon^{\frac{1}{2}})$. When $\mu = O(\epsilon^{\frac{2}{3}})$ radiation would be the dominant attenuating mechanism, yielding $f = O(\epsilon^{\frac{1}{3}})$. In Lettau's experiments $l/L \simeq 3\times 10^{-3}$ (l is the amplitude of the piston displacement) and this is an order of magnitude less than in Sturtevant's experiments ($l/L \approx 1{\cdot}4\times 10^{-2}$). The fact that shocks were observed by Lettau at the open end only at a multiple of the fundamental frequency, and not at the fundamental, is further evidence of the finite rate effect which has been carefully discussed in I.

Van Wijngaarden observed the maximum response amplitude at a frequency less than the linear resonant frequency. In §4 we noted that the nonlinear interaction of the waves gives rise to the same qualitative feature. Nevertheless we must be circumspect in ascribing the frequency shift solely to the nonlinear interaction. For example, in the experiment reported by van Wijngaarden the frequency shift is $O(\epsilon^{\frac{1}{2}})$, whereas the frequency shift due to nonlinear interaction is $O(\epsilon)$. For these experimental conditions the theory of Levine & Schwinger (1948), in which the radial distribution of radiated energy is accounted for by a 'correction' to the length of the pipe, gives the observed order of magnitude of the frequency shift (of order R/L, where R is the radius of the pipe). In Sturtevant's experiments $\epsilon \simeq 1{\cdot}4\times 10^{-2}$ while $R/L \simeq 10^{-2}$, indicating that for $f \sim O(\epsilon^{\frac{1}{3}})$ the nonlinear shift could be dominant. This could, perhaps, be determined more definitively by further experiments. If it were dominant, then the practice of determining the 'length' of the column of gas by adjusting the frequency to obtain the best fit of experimental and theoretical results and using the results of Levine & Schwinger to calculate L would be suspect.

The results in this paper were obtained in the course of research sponsored in part by the U.S. Army under Contract DAAD05-71-C-0389 and monitored by the Ballistics Research Laboratories, Aberdeen Proving Ground, Maryland, and in part by the Army Research Office at Durham under grant DA-ARO-D-31-124-72-G47 to New York University.

Appendix A

Here we prove the existence of a continuous periodic solution of (5.1) with zero mean for $\delta < \delta_c$. If such a curve exists it is obviously unique.

By definition, when $\delta < \delta_c$, $G_m/\mu < \phi$. Consider an integral curve $Z_1(\eta)$ such that $G_m/\mu < Z_1(\eta_1) < \phi$ for some η_1. Then, by (5.1), $Z_1'(\eta_1) < 0$ and, since $Z = G(\eta)/\mu$ is the isocline $Z' = 0$,

$$-\phi < -G_m/\mu < Z_1(\mu) < \phi \quad \text{for all} \quad \eta < \eta_1. \tag{A 1}$$

Thus, since $Z = \pm\phi$ are the isoclines $|Z'| = \infty$, $Z_1'(\eta)$ is bounded and $Z_1(\eta)$ continuous for $\eta < \eta_1$. Letting $\eta_0 = \eta_1 - n$ $(n = 1, 2, \ldots)$ and using condition (2.6), (5.2) may be written as

$$\tfrac{1}{3}nb(1+\delta)\,[Z_1(\eta_1) - Z_1(\eta_0)]\,\{Z_1^2(\eta_1) - \phi^2 + Z_1^2(\eta_0) - \phi^2 + Z_1(\eta_1)\,Z_1(\eta_0) - \phi^2\} = -\mu\int_{\eta_0}^{\eta_1} Z_1(s)\,ds. \tag{A 2}$$

Equations (A 1) and (A 2) then imply that

$$\int_{\eta_0}^{\eta_1} Z_1(s)\,ds > 0.$$

By considering an integral curve $Z_2(\eta)$, such that $-\phi < Z_2(\eta_1) < -G_m/\mu$, over the interval (η_0, η_1) a similar argument shows that $Z_2(\eta)$ is continuous and satisfies

$$\int_{\eta_0}^{\eta_1} Z_2(s)\,ds < 0.$$

Since all integral curves between Z_1 and Z_2 are continuous in (η_0, η_1), there exists a curve $Z(\eta)$ such that

$$\int_{\eta_0}^{\eta_1} Z(s)\,ds = 0.$$

Equation (5.2) then implies that $Z(\eta_1) = Z(\eta_1 - n)$ $(n = 1, 2, \ldots)$. $Z(\eta)$ is continuous and periodic with zero mean value.

Appendix B

We prove the result used in §5: a sufficient condition for the existence of a continuous periodic solution of (5.1) for a given $\delta < -(a+\epsilon_1)$, for any $\epsilon_1 > 0$, is that

$$I^+(\eta) = \mu^2 + 2BG'(\eta)\phi > 0 \quad \text{for} \quad A_0 < \eta < A_2 \tag{B 1}$$

and

$$I^-(\eta) = \mu^2 - 2BG'(\eta)\phi > 0 \quad \text{for} \quad A_2 < \eta < A_4, \tag{B 2}$$

where $B = 4nb(1+\delta)$.

We wish to prove that the separatrices leaving the saddle points at A_i are continuous, single valued and end at the adjacent node. We shall show that, under condition (B 1), the separatrix Z_0^+ leaving A_0 at (θ_0, ϕ) reaches B_1, located at (θ_1, ϕ), $\theta_0 < \theta_1$. The other proofs are similar.

First, since the curve $G(\eta)/\mu$ is the isocline $Z'(\eta) = 0$, and $Z(\eta) \downarrow \phi$ yields the isocline $Z'(\eta) \to +\infty$, the separatrix Z_0^+ is continuous and differentiable in (θ_0, θ_1) and satisfies

$$(\epsilon_1/b)^{\frac{1}{2}} < \phi < Z_0^+(\eta) \leqslant \max_{\eta}\{G(\eta)/\mu\}.$$

In particular, $Z_0^+(\theta_1) \geqslant \phi$. We show that $Z_0^+(\theta_1) = \phi$ by bounding Z_0^+ above by a function $Y(\eta)$ which has the properties

$$Y(\eta) > \phi \quad \text{for} \quad \theta_0 < \eta < \theta \quad \text{and} \quad Y(\theta_0) = Y(\theta_1) = \phi.$$

Such a curve bounds Z_0^+ if $dZ_0^+/d\eta < Y'(\eta)$, for all η in $[\theta_0, \theta_1]$, whenever $Z_0^+ = Y$. The curve $Y(\eta) = 2G(\eta)/\mu - \phi$ has these properties whenever (B 1) holds. For, since when $\delta < -(a+\epsilon_1)$ (5.1) may be written as

$$Z' = 4(G - \mu Z)/B(Z-\phi)(Z+\phi),$$

when $Z_0^+ = Y$, $dZ_0^+/d\eta = -\mu^2/BG$. But $(\epsilon_1/b)^{\frac{1}{2}} < \phi < G(\eta)/\mu$ for $\theta_0 < \eta < \theta_1$, and thus

$$\frac{dZ_0^+}{d\eta} < -\frac{\mu}{B\phi} < \frac{2G'(\eta)}{\mu} = Y'(\eta)$$

whenever (B 1) holds. Hence the result.

Note added in proof. Recently, Jimenez (1973) used the adaption of Lin's technique given in Mortell (1971) to discuss resonant motions in both closed and open pipes for the special piston motion $H(\eta) = \sin 2\pi\eta$. The idea of characterizing an interface in a *nonlinear* problem by a reflexion (or impedance) coefficient was given in Mortell & Varley (1970), and was used in Mortell & Seymour (1972, 1973) and in I. This model is also used by Jimenez. The analytical results given here and in I contain all the results of the special case considered by Jimenez and additionally confirm the results of his extensive numerical work. A major difference between our work and that of other authors is our recognition of the *small rate limit* (see §2 of I and §4 here) and its inherent limitations. The ordinary differential equations which are invariably used to determine the shape of the signal are valid only in the small rate limit when signals travel undistorted as acoustic waves. Then, if the pressure signal on the piston is continuous, it is continuous everywhere in the pipe. A *finite rate* theory must be used to explain an observation that the pressure on the piston is continuous, but shocks may appear at the open end (see §6). The assumption that a resonant motion lies in the neighbourhood of a linear standing wave (Mortell (1971) or equivalently equation (3.21) of Jimenez (1973)) is again valid only in the small rate limit. A full discussion of a finite rate theory, which necessitates an analysis of a functional difference equation, is in preparation.

REFERENCES

COLLINS, W. D. 1971 Forced oscillations of systems governed by one-dimensional nonlinear wave equations. *Q. J. Mech. Appl. Math.* **24**, 129–153.

JIMENEZ, J. 1973 Nonlinear gas oscillations in pipes. Part 1. Theory. *J. Fluid Mech.* **59**, 23–46.

LETTAU, E. 1939 *Dtsch. Kraftfahrtforsch.* **39**, 1.

LEVINE, H. & SCHWINGER, J. 1948 On the radiation of sound from an unflanged circular pipe. *Phys. Rev.* **73**, 383–406.

LIN, C. C. 1954 On a perturbation method based on the method of characteristics. *J. Math. Phys.* **33**, 117–134.

MORTELL, M. P. 1971 Resonant thermal-acoustic oscillations. *Int. J. Engng Sci.* **9**, 175–192.

MORTELL, M. P. & SEYMOUR, B. R. 1972 Pulse propagation in a nonlinear viscoelastic rod of finite length. *SIAM J. App. Math.* **22**, 209–224.

MORTELL, M. P. & SEYMOUR, B. R. 1973 The evolution of a self-sustained oscillation in a nonlinear continuous system. *J. Appl. Mech., Trans. A.S.M.E.* **95**, 53–60.

MORTELL, M. P. & VARLEY, E. 1970 Finite amplitude waves in a bounded media: nonlinear free vibrations of an elastic panel. *Proc. Roy. Soc.* A **318**, 169–196.

SEYMOUR, B. R. & MORTELL, M. P. 1973 Resonant acoustic oscillations with damping: small rate theory. *J. Fluid Mech.* **58**, 353–374.

VAN WIJNGAARDEN, L. 1968. On the oscillations near and at resonance in open pipes. *J. Engng Math.* **2**, 225–240.